Let $b=7$. Then $z=1$ so $a=7-\dfrac{1}{2}=\dfrac{13}{2}$ and $=7+\dfrac{1}{2}=\dfrac{15}{2}$.

Here we have the triangle $\left(\dfrac{13}{2},\dfrac{14}{2},\dfrac{15}{2}\right)$ with rational, non-integer sides with $A=P=21$.

Example 3: Note that the previous example used $b=P/3$. This is always a valid value for b. In this case, we have the triangle
$\left(\dfrac{P}{3}-\dfrac{1}{6}\sqrt{P^2-432},\dfrac{P}{3},\dfrac{P}{3}+\dfrac{1}{6}\sqrt{P^2-432}\right)$
which has $A=P$. This triangle has rational sides only for $P=21,24,31,39,56$ and 109.

Also solved by Kee-Wai Lau, Hong Kong, China David E. Manes, Oneonta, NY; Albert Stadler, Herrliberg, Switzerland, and the proposer

5477: *Proposed by Daniel Sitaru, "Theodor Costescu" National Economic College, Drobeta Turnu-Sevrin, Meredinti, Romania*

Compute:

$$L=\lim_{n\to\infty}\left(\ln n+\lim_{x\to 0}\frac{1-\sqrt{1+x^2}\,\sqrt[3]{1+x^2}\cdot\ldots\cdot\sqrt[n]{1+x^2}}{x^2}\right).$$

Solution 1 by Ed Gray, Highland Beach, FL

We rewrite the expression as:

1. $\displaystyle\lim_{x\to 0}\frac{\left[1-(1+x^2)^{\frac{1}{2}+\frac{1}{3}+\frac{1}{n}+\ldots+\frac{1}{n}}\right]}{x^2}$.

2. Let $N=\displaystyle\sum_{k=2}^{k=n}\frac{1}{k}$, i.e., the harmonic series -1

3. Now consider $\displaystyle\lim_{x\to 0}\frac{\left[1-(1+x^2)^N\right]}{x^2}$.

We expand $(1+x^2)^N$ by the Binomial Theorem:

4. $(1+x^2)^N=1+Nx^2+\dfrac{N(N-1)}{2!}x^4+\ldots$

Then

5. $\displaystyle\lim_{x\to 0}\frac{\left[1-\left(1+Nx^2+\frac{N(N-1)}{2}x^4+\ldots\right)\right]}{x^2}$, or

6. $\displaystyle\lim_{x\to 0}\frac{\left[-Nx^2+\frac{-N(N-1)}{2}x^4+\ldots\right]}{x^2}=\frac{-Nx^2}{x^2}=-N.$

The original limit becomes

7. $\displaystyle\lim_{n\to\infty}(\ln(n)-N)=\lim_{n\to\infty}\left(\ln(n)-\sum_{k=2}^{k=n}\frac{1}{k}\right)=\lim_{n\to\infty}(\ln(n)+1-\text{Harmonic series}).$

The Euler-Mascheroni Constant is defined as $\gamma=\lim_{n\to\infty}\textit{The Harmonic series}-\ln(n)$.
Therefore our expression in step 7 equals $1-\gamma$.

Solution 2 by Bruno Salgueiro Fanego, Viveiro, Spain

Since

$$\lim_{x\to 0}\frac{1-\left(1+x^2\right)^{\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n}}}{x^2}=\lim_{x\to 0}\frac{1-\left(1+x^2\right)^{H_n-1}}{x^2}\left[\frac{0}{0}=Indet.\right]\underset{L'Hospital}{=}$$

$$\lim_{x\to 0}\frac{0-(H_n-1)\,(1+x)^{H_n-2}\,2x}{2x}=(1-H_n)\lim_{x\to 0}\left(1+x^2\right)^{H_n-2}=1-H_n,$$

$$L=\lim_{n\to\infty}(\ln n+1-H_n)=1-\lim_{n\to\infty}(H_n-\ln n)=1-\gamma,$$

where H_n is the n-th harmonic number and γ is the Euler-Mascheroni constant.

Solution 3 by Paolo Perfetti, Department of Mathematics, Tor Vergata University, Rome, Italy

$$\sqrt{1+x^2}\,\sqrt[3]{1+x^2}\cdots\sqrt[n]{1+x^2}=(1+x^2)^{\frac{1}{2}+\frac{1}{3}+\frac{1}{n}}=(1+x^2)^{H_n-1}$$

We have

$$\lim_{n\to\infty}\left(\ln n+\lim_{x\to 0}\frac{1-(1+x^2)^{H_n-1}}{x^2}\right)$$

Now

$$\lim_{x\to 0}\frac{1-(1+x^2)^{H_n-1}}{x^2}=-H_n+1$$

thus

$$L=\lim_{n\to\infty}(\ln-\ln n-\gamma+o(1)+1)=-\gamma+1$$

Solution 4 by Julio Cesar Mohnsam and Mateus Gomes Lucas, both from IFSUL, Campus Pelots-RS, Brazil, and Ricardo Capiberibe Nunes of E.E. Amlio de Caravalho Bas, Campo Grande-MS, Brazil

$$L=\lim_{n\to\infty}\left(\ln n+\lim_{x\to 0}\frac{1-(1+x^2)^{H_n-1}}{x^2}\right)=\lim_{n\to\infty}\left(\ln n+\lim_{x\to 0}(1-H_n)(1+x^2)^{H_n-2}\right).$$

because,

$$\lim_{x\to 0}\frac{[1-(1+x^2)^{H_n-1}]}{(x^2)}\overset{\frac{0}{0}}{=}\lim_{x\to 0}\frac{[1-(1+x^2)^{H_n-1}]'}{(x^2)'}=\lim_{x\to 0}(-H_n+1)(1+x^2)^{H_n-2}=-H_n+1$$

Therefore:

$$L=\lim_{n\to\infty}(\ln n-H_n+1)=\lim_{n\to\infty}(\ln n-H_n)+1=1+\lim_{n\to\infty}(\ln n-H_n)=1-\gamma$$

Note: γ is Euler-Mascheroni constant and $H_n=1+\frac{1}{2}+\cdots+\frac{1}{n}$.

Also solved by Yen Tung Chung, Taichung, Taiwan; Serban George Florin, Romania; Kee-Wai Lau, Hong Kong, China; Moti Levy, Rehovot, Israel;